ETF2700/ETF5970 Mathematics for Business

Lecture 5

Monash Business School, Monash University,
Australia

Outline

Last week:

- Non-linear functions
- Differentiation

This week:

- Increasing/decreasing and convex/concave functions
- Single-variable optimization
- Linear and quadratic approximations
- Elasticity

Last week example: A monopoly compay

Suppose that your company has a monopoly advantage on the market.

- You can determine the market price (in \$k) $P \in (0, 20)$
- The market demand (in thousands) is $Q = 100 - 5P$

The total revenue function is

$$f(P) = P \cdot Q(P) = 100P - 5P^2, \quad P \in (0, 20),$$

and its derivative is $f'(P) = 100 - 10P$

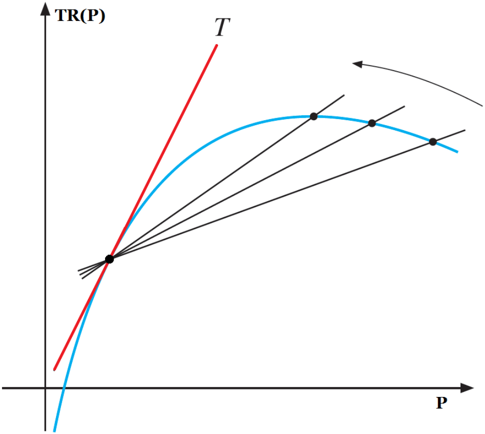
How can we derive $f'(P)$?

1) Calculate $\frac{f(P+\Delta)-f(P)}{\Delta} = 100 - 10P + 5\Delta$
2) Plug in $\Delta = 0$ to get $f'(P) = 100 - 10P$

Power functions and arithmetic rules

1) Write $f(P) = 100f_1(P) - 5f_2(P)$ with $f_1(P) = P$ & $f_2(P) = P^2$
2) $f'(P) = 100 \cdot f_1'(P) - 5 \cdot f_2'(P) = 100 \cdot 1 - 5 \cdot 2P = 100 - 10P$

The derivative (function) is defined as the slope (function) of the tangent line at P



When P changes, the tangent line will change, and so will the slope of the tangent line.

The tangent line: A linear function

1) The tangent line for $f(P)$ at point $P = 5$

$$L(p) = m \cdot p + c,$$

where m is the slope of the tangent line, is actually $f'(p)$ computed at $p = 5$. Thus, the slope of the tagent line is

$$m = f'(5) = 100 - 10 \cdot 5 = 50.$$

2) What is the value of c? Note that $f(5) = 375$. So the point $(5, 375)$ is on the tangent line.
3) Therefore, we have $375 = 50 \cdot 5 + c$, which leads to $c = 125$. So, the tangent line at $P = 5$ is

$$L(p) = 50p + 125.$$

Tangent line of a function: General

- Consider a general function $f(x)$ with derivative $f'(x)$
- The tangent line for $f(x)$ at the point $(a, f(a))$ is

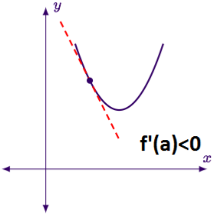
$$L_a(x) = f'(a) \cdot x + c$$

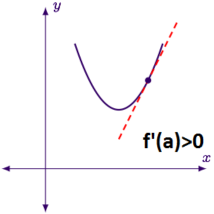
where $c = f(a) - f'(a) \cdot a$ due to the fact that the point $(a, f(a))$ is located on the tangent line.

- You may also write

$$L_a(x) = f'(a)(x - a) + f(a)$$

- $f'(a) > 0$: $L_a(x)$ increases with x
- $f'(a) < 0$: $L_a(x)$ decreases with x





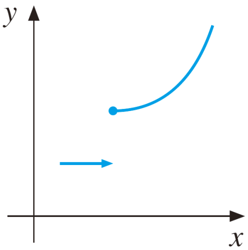
An increasing or decreasing function

If $f'(x) \geq 0$ (or $f'(x) \leq 0$) for all $x \in (a, b)$, then

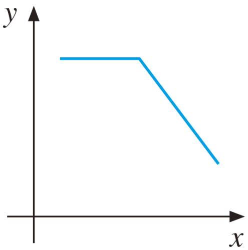
$$f(x) \text{ is increasing (decreasing) in } (a, b),$$

that is, for all $x_1, x_2 \in (a, b)$

$$x_1 < x_2 \quad \rightarrow \quad f(x_1) \leq f(x_2) \quad (\text{or } f(x_1) \geq f(x_2))$$



Increasing



Decreasing

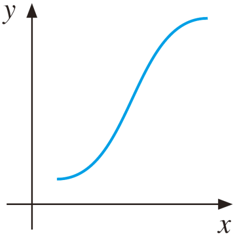
Strictly increasing or decreasing

If $f'(x) > 0$ (or $f'(x) < 0$) for all $x \in (a, b)$, then

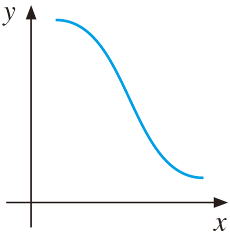
$$f(x) \text{ is } \textbf{strictly} \text{ increasing (decreasing) in } (a, b),$$

that is, for all $x_1, x_2 \in (a, b)$

$$x_1 < x_2 \quad \Leftrightarrow \quad f(x_1) < f(x_2) \quad (\text{or } f(x_1) > f(x_2))$$



Strictly increasing



Strictly decreasing

Example of a monopoly company

The derivative of the total revenue function

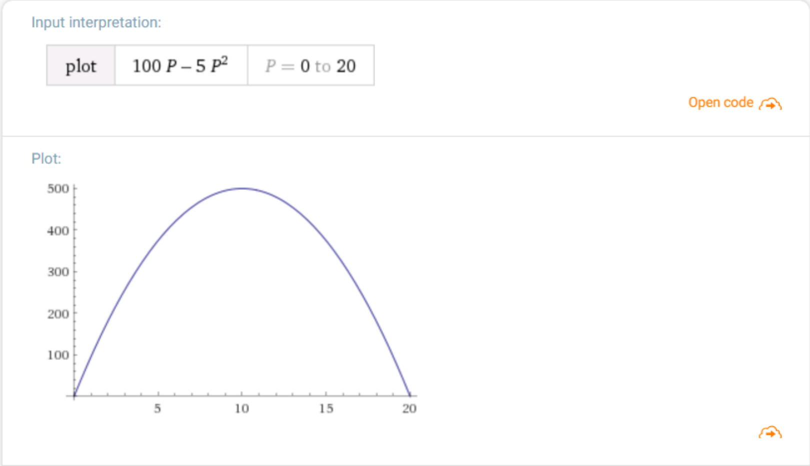
$$f'(P) = 100 - 10P$$

- $f'(P) = 0$ when $P = 10$
- $f'(P) > 0$ when $P \in (0, 10)$

  therefore, $f(P)$ is strictly increasing in $(0, 10)$
- $f'(P) < 0$ when $P \in (10, 20)$

  therefore, $f(P)$ is strictly decreasing in $(10, 20)$

Maximum total revenue at $P = 10$?



Stationary point

A stationary point x is a point, at which $f'(x) = 0$.

In our example of total revenue:

- $f'(10) = 0$, thus $P = 10$ is a stationary point, and is also the maximum point.
- Sometimes we cannot find a stationary point:

$$\text{for example,}\quad f(x) = e^x, \quad f'(x) = e^x > 0$$

Stationary point: Quadratic functions

- Consider a quadratic function $(a \neq 0)$

$$f(x) = ax^2 + bx + c, \quad x \in D$$

- We know its derivative is $f'(x) = 2ax + b$
- Solve $2ax + b = 0$ to get $x = -\frac{b}{2a}$.
- If $-\frac{b}{2a} \in D$, it is the stationary point; Otherwise there is no stationary point.

First-order condition

A function $f(x)$ on (a, b) with derivative $f'(x)$.

- Any maximum/minimum point $c \in (a, b)$ satisfies $f'(c) = 0$
- Maximum/Minimum point must be a stationary point
- But in general, a stationary point may not be a maximum/minimum point

How to find a maximum or minimum point

If there is a maximum point in (a, b):

- Solve $f'(x) = 0$ to find all stationary points in (a, b).
- If only one point is found, then it is the max/min point.
- If multiple points are found, then we need to compare the $f(\cdot)$ values and take the one(s) with largest $f(\cdot)$ value.
- A similar procedure applies to locating the minimum point.

Example

Determine the minimal value of the function

$$f(x) = x^3 - 12x, \quad x \in (0, 5).$$

We may assume that the minimal point exists.

1) Determine the derivative $f'(x) = 3x^2 - 12$
2) Solve $3x^2 - 12 = 0$ by the 'abc' method to get
$$x_1 = 2, \quad x_2 = -2 \text{ (not in the domain)}$$
3) The minimal value is
$$f(2) = 2^3 - 12 \times 2 = -16$$

When will stationarity imply optimality?

- The stationary points in our examples so far are all maximum/minimum points.
- This is NOT true in general. For example, $f(x) = x^3$, $x \in (-\infty, \infty)$ has only stationarity point $x = 0$, but it is not a maximum or a minimum point.
- However, stationarity implies optimality for concave and convex functions.

What is a concave/convex function? Example of TR

Suppose now your company monopolies two identical markets and you can determine the market prices $P_1 \in (0, 20)$ and $P_2 \in (0, 20)$ in both markets.

- Market demand in the markets:
  $Q_1 = 100 - 5P_1$, $Q_2 = 100 - 5P_2$
- Total Revenue in both markets:
  $TR_1(P_1) = f(P_1), \quad TR_2(P_2) = f(P_2)$, where
  $f(P) = 100P - 5P^2$. To maximize TR, shall we set $P_1 = P_2$?

Example

Compare the following pricing strategies:

1) $P_1 = 6$, $P_2 = 10$
2) $P_1 = P_2 = (6 + 10)/2 = 8$

Which gives a larger total revenue from both markets?

1) $TR = f(6) + f(10) = 420 + 500 = 920$
2) $TR = 2 \cdot f(8) = 2 \cdot 480 = 960 > 920$

The second pricing strategy gives a larger total revenue.

Averaged price is better

In fact, we can show

$$2 \cdot f\left(\frac{P_1 + P_2}{2}\right) \geq f(P_1) + f(P_2),$$

for **all** $P_1, P_2 \in (0, 20)$.

In other words, using an averaged price in the two markets always gives higher total revenue.

Yes, we should set $P_1 = P_2$.

Concave function: Middle point is better

A continuous function $f(x)$, $x \in D$ is concave if

$$f\left(\frac{x_1 + x_2}{2}\right) \geq \frac{1}{2}\left(f(x_1) + f(x_2)\right)$$

for **all** $x_1, x_2 \in D$.

- Example: $f(P) = 100P - 5P^2$, $P \in (0, 20)$ is concave

Convex function: Middle point is worse

A continuous function $f(x)$, $x \in D$ is convex if

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{1}{2}\left(f(x_1) + f(x_2)\right)$$

for **all** $x_1, x_2 \in D$.

- Example: $f(x) = x^3 - 12x$, $x \in (0, 5)$ is convex.

Stationarity, and concavity/convexity

- A stationary point for a concave/convex function $f(x)$ on (a, b) is a maximum/minimum point for $f(x)$ on (a, b).
- In other words, for $f(x)$ defined on (a, b)

Stationarity & Concavity/Convexity =
Maximum/Minimum

- How can we check whether a function is concave/convex or not?

Derivative of a derivative function

Consider a function f defined on (a, b)

- $f''(x) \leq 0$ for **all** $x \in (a, b)$: f is concave
- $f''(x) \geq 0$ for **all** $x \in (a, b)$: f is convex

Here, $f''(x)$ is the derivative of $f'(x)$, also known as second order derivative of $f(x)$.

- $f''(x)$ is called the second derivative of f.
- $f'(x)$ may be called the first derivative of f.

Example: Second (order) derivative

Let's consider our total revenue function

$$f(P) = 100P - 5P^2, \quad P \in (0, 20)$$

1) (First) derivative:

$$f'(P) = 100 - 10P, \quad P \in (0, 20)$$

2) Second (order) derivative:

$$f''(P) = 0 - 10 \cdot 1 = -10 < 0, \quad P \in (0, 20)$$

Therefore, $f(P)$ is concave.

Another example: Second derivative

Recall our another example

$$f(x) = x^3 - 12x, \quad x \in (0, 5)$$

1) (First) derivative:

$$f'(x) = 3x^2 - 12, \quad x \in (0, 5)$$

2) Second (order) derivative:

$$f''(x) = 6x > 0, \quad x \in (0, 5)$$

Hence, $f(x)$ is convex.

Second derivative: Quadratic functions

Consider a quadratic functions

$$f(x) = ax^2 + bx + c, \quad x \in D$$

1) (First-order) derivative is $f'(x) = 2ax + b$
2) Second (order) derivative is $f''(x) = 2a$

i) $a < 0$: $f(x)$ is concave (the graph of parabola looks like a cap)
ii) $a > 0$: $f(x)$ is convex (the graph of parabola looks like a cup or U shape)

Optimisation: Regular cases

A function $f(x)$ for $x \in (a, b)$ with derivative $f'(x)$.

If we know there is a maximum/minimum point

1) Solve $f'(x) = 0$ to find all stationary points in (a, b)
2) Compare f values at stationary points take the one with largest/smallest value.

If we don't know if there is a maximum/minimum point, but find $f(x)$ is convex/concave

i) Concave: a stationary point is a maximum point
ii) Convex: a stationary point is a minimum point

Optimisation: Irregular cases*

A continuous function $f(x)$ for $x \in (a, b)$ with derivative $f'(x)$.

- $f(x)$ is not convex or concave
- Don't know if there is a max/min point
- Still compare f values at stationary points
- May use the sign of $f'(x)$ to determine how the function increases and decreases in different intervals to determine whether a stationary point is a max/min point, or neither.

Optimisation in a closed interval

Consider a function $f(x)$ on $[a, b]$. The optimal value of $f(x)$, if it exists, can only be achieved at

- the end-point(s) a or/and b
- or/and in the interior (a, b)

A two-step procedure

If we know there is a maximum/minimum point:

- determine the stationary points in (a, b)
- evaluate the $f(\cdot)$ values at stationary points
- compare with the end-point values $f(a)$ and $f(b)$

Example

Determine the maximum value of the function

$$f(x) = x^2 - x, \quad x \in [0, 20]$$

Assume there is a maximum value (that is, can be achieved)

1) Solve $f'(x) = 0$, in other words,

   $2x - 1 = 0$ to get $x = 1/2 = 0.5$ (insidethe domain)
2) $f(1/2) = (1/2)^2 - 1/2 = -1/4$
3) $f(0) = 0$ and $f(20) = 20^2 - 20 = 380$.

Maximum value is 380.

Example

Determine the maximal value of the function

$$f(x) = x^2 + x, \quad x \in [0, 20]$$

Assume there is a maximum value (that is, can be achieved)

1) Solve $f'(x) = 0$, in other words,

   $$2x + 1 = 0 \text{ to get } x = -0.5 \text{ (outside the domain)}$$

2) No stationary point, so no optimal value in $(0, 20)$
3) $f(0) = 0$ and $f(20) = 20^2 + 20 = 420$.

Maximum value is 420.

Existence of an optimal value

Extreme value theorem

If $f(x)$ is continuous in $[a, b]$, then $f(x)$ has a minimum point x_1 and a maximum point x_2 both in $[a, b]$ so that

$$f(x_1) \leq f(x) \leq f(x_2), \quad \text{for all } x \in [a, b].$$

What is a continuous function?

Continuous function

- $f(x)$ is continuous if ‘you could draw its graph without lifting your pen from the paper’

If $f'(x)$ is well defined on $[a, b]$, then $f(x)$ is continuous

- Polynomial, exponential, logarithm functions are continuous (when defined properly), and so are their sums, differences, products and divisions.
- Almost all functions considered in this unit are continuous.

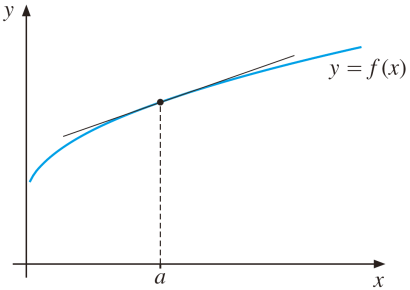
Approximations with Derivatives

Linear approximation

The *linear approximation* to $f(x)$ at $x = a$ is

$$f(x) \approx L_a(x) = f'(a)(x - a) + f(a), \quad x \approx a.$$

Here $L_a(x)$ is the tangent line at the point $(a, f(a))$.



Use linear approximation

If we treat $x + \Delta \approx x$, we may approximate

$$
\begin{aligned}
f(x+\Delta) \approx L_x(x+\Delta) =& f'(x)(x+\Delta-x)+f(x) \\
=& f'(x)\Delta + f(x),
\end{aligned}
$$

Subtract $f(x)$ on both sides,

$$f(x+\Delta) - f(x) \approx f'(x)\Delta$$

If we divide both side of the equation by Δ, then

$$\frac{f(x+\Delta)-f(x)}{\Delta} \approx f'(x),$$

which is the definition of derivative.

Total revenue example

The total revenue of the company is

$$f(P) = 100P - 5P^2$$

Allow $\Delta = 1$, then

$$\begin{aligned} f(P+1) - f(P) &\approx f'(P) \\ &= 100 - 10P. \end{aligned}$$

For example, when $P = 5$, we have $f'(5) = 50$

- When P increases by 1 thousand dollars from 5 thousand dollars, the total revenue will increase approximately by 50 million dollars.
- The actual increment is $f(6) - f(5) = 45$

Last week's example: Percentage change

Recall our revenue $f(P) = 100P - 5P^2$, $P \in (0, 20)$.

- $P = 8$: increase by 1% from 8 to 8.08

Question

What is percentage change in f approximately?

- Change in $f(P)$:

$$f(8.08) - f(8) \approx f'(8) \cdot (8.08 - 8) = 20 \cdot 0.08 = 1.6$$

- Percentage change in $f(P)$ is

$$\frac{f(8.08) - f(8)}{f(8)} \times 100\% \approx \frac{1.6}{480} \times 100\% \approx 0.333\%$$

- The exact percentage change is 0.327%

Quadratic approximation

The *quadratic approximation* to $f(x)$ at $x = a$ is

$$f(x) \approx Q_a(x) = \frac{1}{2}f''(a)(x-a)^2 + f'(a)(x-a) + f(a)$$

for $x \approx a$.

The '$\approx$' sign can be replaced by '$=$' sign if $f(x)$ is a quadratic function itself.

Use quadratic approximations

If we treat $x + \Delta \approx x$, we can approximate

$$\begin{aligned} f(x+\Delta) \approx & Q_x(x+\Delta) \\ = & \frac{1}{2} f''(x)(x+\Delta-x)^2 + f'(x)(x+\Delta-x) \\ & + f(x), \end{aligned}$$

Subtract $f(x)$ from both sides:

$$f(x+\Delta) - f(x) \approx \tfrac{1}{2} f''(x)\Delta^2 + f'(x)\Delta$$

Example

Let $f(x) = e^x, x \in (0, 10)$. Approximate $f(5) - f(3)$ by

- linear approximation
- quadratic approximation

at $x = 3$.

We know $f'(x) = e^x$ and therefore $f''(x) = e^x$.

- $f(3+2) - f(3) \approx f'(3) \times 2 = e^3 \times 2 \approx 40.17$
- $f(3+2) - f(3) \approx \frac{1}{2} f''(3) \cdot 2^2 + f'(3) \cdot 2 = \frac{1}{2} \cdot e^3 \cdot 4 + e^3 \cdot 2 = 4 \cdot e^3 \approx 80.34$
- True increment: $f(5) - f(3) = 128.33$