ETF2700/ETF5970 Mathematics for Business

Lecture 4

Monash Business School, Monash University, Australia

Outline

Last week:

- Matrix
- Matrix operation, and inverse matrix
- Eigenvalues and eigenvectors
- Linear programming

This week:

- Non-linear functions
- Differentiation

Relation between variables

- x: input real-value variable
- y: output real-value variable
- Relationship between y and x is expressed as

$$y = f(x), \qquad x \in D$$

where D is a set of all possible input values, and $f(x)$ is the real-value output assigned to each real-value input $x \in D$

Linear function

For some known real values m and c

$$f(x) = mx + c, \qquad x \in D$$

where

- $m = f(x+1) - f(x)$ is called *slope* (such as 'variable cost')
- $c = f(0)$ is called *intercept* (such as 'fixed cost')

Is there other type of functions? Yes

An example: Monopoly company

- Suppose you own the only company in the market, and you can determine the market price $P \in (0, 20)$
- The market demand (your company's sales) is given by

$$Q = 100 - 5P$$

- What is your total revenue given the market price P?

Revenue Function

- Market price P: an input variable
- Total revenue TR: an output variable
- The total revenue as a function of price is

$$TR = f(P), \quad P \in (0, 20)$$

which we assume to be a quadratic (non-linear) function:

$$f(P) = PQ = P(100 - 5P) = 100P - 5P^2$$

A quadratic function in general

- A quadratic function in x is of the form

$$f(x) = ax^2 + bx + c, \text{ for } x \in D, \text{ with } a \neq 0$$

- In our example: A quadratic function in P is given by

$$f(P) = 100P - 5P^2, \quad P \in (0, 20)$$

with $a = -5$, $b = 100$ and $c = 0$ (to use the abc formula)

"Partition" of functions

- This quadratic function can be written as **weighted sum** of basic functions

$$f(P) = 100 \cdot f_1(P) + (-5) \cdot f_2(P), \quad P \in (0, 20)$$

where $f_1(P) = P$ and $f_2(P) = P^2$, which are called the *power* functions

Power functions as “building” blocks

A power function at a known order k is of the form

$$f_k(x) = x^k, \quad x \in D.$$

Example: $f_2(x) = x^2, f_1(x) = x^1, f_0(x) = 1$ (even if $x = 0$) and $f_{-1}(x) = x^{-1}$.

“Partition” of the quadratic functions

$$f(x) = ax^2 + bx + c = a \cdot f_2(x) + b \cdot f_1(x) + c \cdot f_0(x)$$

Weighted sum of power functions

- A polynomial function of the order k is of the form:

  $$f(x) = a_k x^k + a_{k-1} x^{k-1} + \ldots + a_0,$$

  which is a weighted sum of power functions.
- It becomes a linear function (for $k = 1$), or a quadratic function (for $k = 2$) or a cubic function (for $k = 3$)

Example

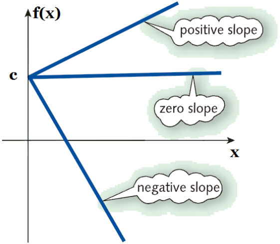
$f(P) = 100P - 5P^2$ is a polynomial (k=2):

$$a_2 = -5, \quad a_1 = 100, \quad a_0 = 0.$$

Slope as a relative change of $f(x)$

The slope of linear $f(x) = mx + c$ is the change of $f(x)$ for a unit change in x. In general, we have

$$m = \frac{f(x + \Delta) - f(x)}{\Delta}, \quad \Delta \neq 0$$



Can we calculate slope of a quadratic function?

- In the linear equation $f(x) = mx + c$, slope is defined as $f(x+1) - f(x)$, or in general, slope of the linear equation is defined as
$$m = \frac{f(x+\Delta) - f(x)}{\Delta}$$
for any value of Δ.
- For quadratic functions such as $f(P) = 100P - 5P^2$, with $P \in (0, 20)$, we can also calculate
$$m = \frac{f(P+\Delta) - f(P)}{\Delta}$$
- However, for different magnitude of Δ and/or at different values of P, the values of m are different.
- $P = 10$ and $\Delta = 1$: $m = -5$
- $P = 10$ and $\Delta = 3$: $m = -15$
- $P = 5$ and $\Delta = 1$: $m = 45$ $P = 5$ and $\Delta = -1$: $m = 55$

Slope of quadratic functions

- Shall we define different slopes at different values of P?
- Shall we define different slopes at different values of Δ?

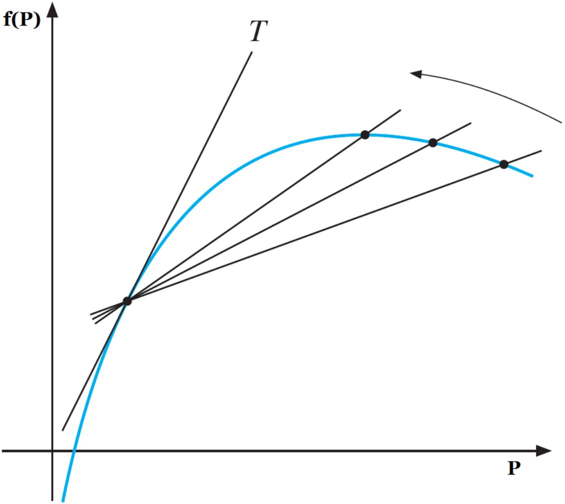
Slopes at different points: Derivative function

For any $P \in (0, 20)$ and $\Delta \approx 0$, we have

$$\begin{aligned}
m =& \frac{f(P+\Delta) - f(P)}{\Delta} \\
=& \frac{\left\{100(P+\Delta) - 5(P+\Delta)^2\right\} - \left(100P - 5P^2\right)}{\Delta} \\
=& \frac{100\Delta - 10P\Delta - 5\Delta^2}{\Delta} \\
=& 100 - 10P - 5\Delta \approx 100 - 10P
\end{aligned}$$

- The derivative of $f(P)$ at point P is $f'(P) = 100 - 10P$.
- The derivative is a lower order polynomial in P

Derivative is the slope of the tangent line of $f(P)$



Derivative: First principle

The derivative of function f at point x is

$$f'(x) = \lim_{\Delta \to 0} \frac{f(x+\Delta) - f(x)}{\Delta}$$

It is obvious that $f'(x)$ is a function in x.

How to compute $f'(x)$

1) Rewrite $\frac{f(x+\Delta)-f(x)}{\Delta}$ and remove Δ in denominator;
2) Plug in $\Delta = 0$ to obtain the derivative function.

Derivative of $f(x) = mx + c$ according to the 1st principle

1) Rewrite

$$\frac{f(x+\Delta) - f(x)}{\Delta} = \frac{(mx + m\Delta + c) - (mx + c)}{\Delta} = \frac{m\Delta}{\Delta} = m$$

2) Plug in $\Delta = 0$ to get the derivative

$$f'(x) = m$$

Derivative of the power function $f_2(x) = x^2$

1) For any x, we rewrite

$$\frac{f_2(x+\Delta) - f_2(x)}{\Delta} = \frac{(x+\Delta)^2 - x^2}{\Delta} = \frac{2\Delta x + \Delta^2}{\Delta} = 2x + \Delta$$

2) Plug in $\Delta = 0$ to get the derivative: $f'(x) = 2x$

Derivative of the power function $f_k(x)) = x^k$

$$f_k'(x) = \begin{cases} k \cdot x^{k-1} & k \neq 0 \\ 0 & k = 0 \end{cases}$$

Examples:

- derivative of $f_3(x) = x^3$ is $f_3'(x) = 3x^2$
- derivative of $f_4(x) = x^4$ is $f_4'(x) = 4x^3$

Sum of "building blocks" of derivatives

Addition Rule

If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.

Example: $f(x) = x + x^2$, for $x \in (-\infty, \infty)$

We can write $f(x) = f_1(x) + f_2(x)$, where $f_1(x) = x$ and $f_2(x) = x^2$. Therefore, we have

$$f'(x) = f_1'(x) + f_2'(x) = 1 + 2x$$

Subtraction Rule

If $f(x) = g(x) - h(x)$, then $f'(x) = g'(x) - h'(x)$.

Example: $f(x) = x - x^2$, for $x \in (-\infty, \infty)$

We can write $f(x) = f_1(x) - f_2(x)$, where $f_1(x) = x$ and $f_2(x) = x^2$. Therefore,

$$f'(x) = f_1'(x) - f_2'(x) = 1 - 2x$$

Multiplication by a constant

If $f(x) = c \cdot g(x)$ for some c, then $f'(x) = c \cdot g'(x)$.

Example: $f(x) = 2x^2$, $x \in (-\infty, \infty)$

We can write $f(x) = 2 \cdot g(x)$, where $g(x) = x^2$. Therefore

$$f'(x) = 2 \cdot g'(x) = 2 \cdot (2x) = 4x$$

Our example: total revenue function

$$f(P) = 100P - 5P^2, \quad \text{for } P \in (0, 20),$$

and its derivative $f'(P) = 100 - 10P$ for $P \in (0, 20)$.
We could write $f(P) = 100 \cdot f_1(P) - 5 \cdot f_2(P)$ with

$$f_1(P) = P, \quad f_2(P) = P^2.$$

Therefore,

$$\begin{aligned} f'(P) &= 100 \cdot f_1'(P) - 5 \cdot f_2'(P) \\ &= 100 \cdot 1 - 5 \cdot 2P = 100 - 10P \end{aligned}$$

Derivative of Quadratic Functions

The derivative of a quadratic function

$$f(x) = ax^2 + bx + c, \quad x \in D$$

is

$$f'(x) = 2ax + b, \quad x \in D.$$

Show this in either way:

1) by definition
2) by the power functions and operation rules

Example: A simple saving problem

Suppose you have \$1000 savings at a bank that incurs interest at 2% annual rate at the end of every year.

- Savings after 1 year: $\$1000 \times (1 + 2\%) = \$1020$
- Savings after 2 years: $\$1020 \times (1 + 2\%) = \$1000 \times (1 + 2\%)^2 = \$1040.40$
- Savings after 3 years: $\$1040.4 \times (1 + 2\%) = \$1000 \times (1 + 2\%)^3 \approx \$1061.21$
- Savings after x years: $\$1000 \times (1 + 2\%)^x$

Exponential Functions

- Number of Years x is the 'input' variable
- Savings S (in **thousand** dollars) is the 'output' variable

Amount of savings S (in thousand dollars) is a function of x:

$$S = f(x), \quad x \in \{1, 2, \ldots\}$$

with $f(x) = (1 + 2\%)^x$, or $f(x) = 1.02^x$, which is an exponential function.

Exponential function

An exponential function is of the form

$$f(x) = a^x, \quad x \in D$$

with some known $a > 0$.

In the above example: an exponential function in x

$$f(x) = 1.02^x, \quad x \in \{1, 2, \dots\}$$

with $a = 1.02$.

Quarterly compounded interest

Suppose the savings of \$1000 still receives 2% interest annually, but by the end of each quarter you will receive $2\% \times \frac{1}{4}$ of the past quarter's interest.

Your savings after

- 1 quarter: $\$1000 \times (1 + \frac{2\%}{4})^{1} = \$1005$
- 2 quarters: $\$1000 \times (1 + \frac{2\%}{4})^{2} \approx \$1010.02$

Your savings after

- 1 **year** (4 quarters): $\$1000 \times (1 + \frac{2\%}{4})^{4} \approx \$1020.15$
- x **years** ($4x$ quarters): $\$1000 \times (1 + \frac{2\%}{4})^{4x}$

Monthly compounded interest

Suppose the savings of \$1000 still receives 2% interest annually, but by the end of each month you will receive $2\% \times \frac{1}{12}$ of the past month's interest.
Your savings after

- after 1 month: $\$1000 \times (1 + \frac{2\%}{12})^{1} \approx \$1001.67$
- after 2 months: $\$1000 \times (1 + \frac{2\%}{12})^{2} \approx \$1003.34$
- after 1 year: $\$1000 \times (1 + \frac{2\%}{12})^{12} \approx \$1020.18$
- x **years** ($4x$ quarters): $\$1000 \times (1 + \frac{2\%}{12})^{12x}$

Suppose the savings of \$1000 still receives 2% interest annually, but the cycle of interest payment m times a year. By the end of each cycle, you will receive $2\% \cdot \frac{1}{m}$ interest of the last cycle.

You savings after

- 1 cycle: $\$1000 \times (1 + \frac{2\%}{m})^1$
- 2 cycles: $\$1000 \times (1 + \frac{2\%}{m})^2$
- **after 1 year** (m cycles): $\$1000 \times (1 + \frac{2\%}{m})^m$
- **after x years** (mx cycles): $\$1000 \times (1 + \frac{2\%}{m})^{mx}$

What is your savings after x years if m is very large?

- It can be shown that

$$\lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m = \mathrm{e}, \quad \text{known as Euler's constant.}$$

- As $m \to \infty$, the effective interest rate over x year is

$$r = \left(1 + \frac{2\%}{m}\right)^{mx} = \left(\left(1 + \frac{0.02}{m}\right)^{(m/0.02)}\right)^{0.02x} \longrightarrow \mathrm{e}^{0.02x}$$

Savings after x years

- Euler's constant $\mathrm{e} \approx 2.7182818$
- Savings after x years is $S = \$1000 \times \mathrm{e}^{0.02x}$
- Savings after 1 year is $\$1000 \times \mathrm{e}^{0.02} \approx 1020.20$

Natural exponential function

$f(x) = e^x$, for $x \in D$.
The solution of $e^x = a$ is denoted as

$$\ln(a), \textit{ the natural logarithm of } a.$$

which is interpreted as $\log_e(a)$, with some textbooks writing it as $\log(a)$.

Natural logarithm function

$f(x) = \ln(x)$, or sometimes $f(x) = \log(x)$, for $x \in D$.
Note that here D **cannot** contain negative values.

Derivatives of natural exponential and log functions

The 1st principle of taking the derivative of $f(x)$ shows that

$$f'(x) = \lim_{\Delta \to 0} \frac{f(x+\Delta) - f(x)}{\Delta}$$

We are able to find out the derivatives of natural exponential and log functions:

- The derivative of $f(x) = e^x$ is $f'(x) = e^x$
- The derivative of $f(x) = \ln(x)$ is $f'(x) = \frac{1}{x}$

Example

Suppose you have a fixed deposit \$1000 savings at bank with annual interest rate 2%. How many years will you deposit it so as to accumulate savings of more than \$1060?

The solution is to solve $1.02^p = 1.06$. According to properties of the exponential function, we have

$p = \log_{1.02}(1.06) = \ln(1.06)/\ln(1.02) \approx 2.94$

Thus, you need to wait at least three years.

Note that $a^p = x$ is equivalent to: $p = \log_a(x)$.

Operation Rules

Taking the quadratic functions as examples, we learned the derivative rules:

- Addition rule
- Subtraction rule
- Product rule with a constant

These rules are actually **applicable to all functions** including exponential and logarithm functions.

Example

If $f(x) = e^x + \ln(x)$, then $f'(x) = e^x + 1/x$

Multiplication by a function

Product Rule

If $f(x) = c(x) \cdot g(x)$, then

$$f'(x) = c'(x) \cdot g(x) + c(x) \cdot g'(x)$$

Example: Let $f(x) = xe^x$ for $x \in (-\infty, \infty)$.

$$c(x) = x, \text{ and } g(x) = e^x.$$

Hence,

$$f'(x) = 1 \cdot e^x + x \cdot e^x = (1 + x)e^x.$$

Product rule: One more example

Define $f_1(x) = x$, $f_2(x) = x^2$, $f_3(x) = x^3$, for $x \in (-\infty, \infty)$.

Use the facts that $f_1'(x) = 1$, $f_2'(x) = 2x$ and the product rule to show that

$$f_3'(x) = 3x^2.$$

Solution: Re-express $f_3(x)$ as $f_3(x) = x \cdot x^2 = f_1(x) \cdot f_2(x)$.
According to the product rule, we have

$$\begin{aligned} f_3'(x) =& f_1'(x) \cdot f_2(x) + f_1(x) \cdot f_2'(x) \\ =& 1 \cdot x^2 + x \cdot 2x = x^2 + 2x^2 = 3x^2 \end{aligned}$$

Dividing by a function

Quotient Rule

If $f(x) = \frac{g(x)}{h(x)}$, then

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{(h(x))^2}$$

Example: let $f(x) = \frac{\ln(x)}{x}$ for $x > 0$.

$$g(x) = \ln(x), \text{ and } h(x) = x.$$

Hence,

$$f'(x) = \frac{\frac{1}{x} \cdot x - \ln x \cdot 1}{(x)^2} = \frac{1 - \ln(x)}{x^2}$$

Function of Functions

Example

Consider $h(x) = x^2$, $g(z) = e^z$, what is $g(h(x))$?

For example, take $x = 2$.

1. Plug in $x = 2$ to obtain $h(2) = 2^2 = 4$.
2. Plug in $z = h(2)$ to get

$$g(h(2)) = g(4) = e^4$$

$$g(h(x)) = g(x^2) = e^{x^2}$$

Chain Rule

If $f(x) = g(h(x))$, then

$$f'(x) = g'(h(x)) \cdot h'(x)$$

Example: $f(x) = e^{x^2}$, that is $g(z) = e^z$ and $h(x) = x^2$

1. Determine $g'(z) = e^z$, so plug in $z = h(x)$ to get

$$g'(h(x)) = e^{h(x)} = e^{x^2}$$

2. Determine $h'(x) = 2x$, so

$$f'(x) = e^{x^2} \cdot (2x) = 2xe^{x^2}$$

Derivative of exponential functions

Define $g(x) = e^x$.

Use the fact that $g'(x) = e^x$ and the chain rule to determine the derivative of $f(x) = a^x$.

Write $f(x) = a^x = \left(e^{\ln(a)}\right)^x = e^{\ln(a)\cdot x} = g(h(x))$ with

$$h(x) = \ln(a) \cdot x,$$

to get

$$\begin{aligned} f'(x) = g'(h(x)) \cdot h'(x) =& g'(\ln(a) \cdot x) \cdot \ln(a) \\ =& e^{\ln(a)\cdot x} \ln(a) = a^x \ln(a). \end{aligned}$$

Derivative in Business

Use derivative for approximations

As $f'(x)$ is defined as

$$\lim_{\Delta \to 0} [f(x+\Delta) - f(x)]/\Delta$$

We can approximate the change in $f(x)$ by

$$f(x+1) - f(x) \approx f'(x)$$

The change in $f(x)$ is **approximately** $f'(x)$ if x increases by $\Delta = 1$.

Example of Monopoly: $f(P) = 100P - 5P^2$, for $P \in (0, 20)$

Recall that $f'(P) = 100 - 10P$, therefore, $f'(8) = 20$.

$$f(9) - f(8) \approx 20$$

However, we know that $f(9) - f(8) = 15$ precisely.

Percentage change

Recall that in our revenue $f(P) = 100P - 5P^2$, for $P \in (0, 20)$.

- Price $P = 8$ increases by 1%: from 8 to 8.08
- What is the percentage change in $f(P)$ approximately?
- Change in $f(P)$:

$$f(8.08) - f(8) \approx f'(8) \cdot (8.08 - 8) = 20 \cdot 0.08 = 1.6$$

- Percentage change in $f(P)$ is

$$\frac{f(8.08) - f(8)}{f(8)} \times 100\% \approx \frac{1.6}{480} \times 100\% \approx 0.333\%$$

Exact percentage change is 0.327%.

Elasticity

If x is changed by 1%, the percentage change in $f(x)$ is

$$\frac{f(x + 1\% \cdot x) - f(x)}{f(x)} \times 100\%$$
$$\approx \frac{f'(x) \cdot (1\% \cdot x)}{f(x)} \times 100\% = \frac{f'(x)x}{f(x)}\%$$

The elasticity of $f(x)$ at point x is

$$\mathrm{El}_x f(x) = \frac{f'(x)x}{f(x)}$$

Elasticity: Revenue Function

Recall our revenue $f(P) = 100P - 5P^2$, $P \in (0, 20)$.

$$\mathrm{El}_P f(P) = \frac{f'(P)P}{f(P)} = \frac{(100 - 10P)P}{100P - 5P^2} = \frac{100P - 10P^2}{100P - 5P^2}$$

Elasticity: Revenue Function

- $P = 8$: $\text{El}_P f(P) = \frac{1}{3} \approx 0.33$
- $P = 10$: $\text{El}_P f(P) = 0$
- $P = 14$: $\text{El}_P f(P) = -1.33$

Elasticity of a power function

Let $f(x) = x^2$, $x \in (-\infty, \infty)$. We have

1) $f'(x) = 2x$
2) $\text{El}_x f(x) = \frac{f'(x)x}{f(x)} = \frac{2x \cdot x}{x^2} = 2$

The elasticity is a constant, which does not depend on x.

Elasticity of the natural exp function

Let $f(x) = e^x$, $x \in (-\infty, \infty)$. We have

1) $f'(x) = e^x$
2) $\text{El}_x f(x) = \frac{f'(x)x}{f(x)} = \frac{e^x \cdot x}{e^x} = x$

Summary

- Non-linear functions: quadratic, polynomial, exponential, natural logarithm
- Derivative definition and operation rules
  - Sum, subtraction, multiplication, and quotient
- Derivative of many basic functions:
  - power, quadratic, exponential and natural logarithm
- Function of functions: chain rule
- Elasticity